Lomonosov's Invariant Subspace Theorem for Multivalued Linear Operators

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ABSTRACT. The famous Lomonosov's invariant subspace theorem states that *if a continuous linear operator T on an infinite-dimensional normed space E "commutes" with a compact operator $K \neq 0$, i.e., $TK = KT$, then T has a non-trivial closed invariant subspace*. We generalize this theorem for multivalued linear operators. We also provide some applications to single-valued linear operators.

1. Introduction.

The Invariant Subspace Problem asks whether every linear operator $h : Y \rightarrow Y$ on an infinite dimensional topological vector space Y has a nontrivial closed *invariant subspace*, i.e., a linear subspace M of Y such that $M \neq \{0\}$, $M \neq Y$ and $h(M) \subset M$ (for a survey and references see [**6**, **1**]). In general the answer is negative and the issue is to investigate the class of operators satisfying this property. It is known that every compact operator k belongs to this class and so does every operator h commuting with k :

$$hk = kh.$$

The famous Lomonosov's invariant subspace theorem [**5**] states the following.

THEOREM 1.1. *Suppose Y is an infinite dimensional normed space, h and k are continuous linear operators, k is compact, nonzero and commutes with h. Then h has a nontrivial closed invariant subspace.*

In this paper we generalize this result for multivalued linear operators. The theory of multivalued linear maps (linear relations) is well developed, see Cross [**2**] A multivalued map $h : X \rightarrow Y$ between vector spaces is called a *linear relation* if

$$h(ax) = ah(x), \quad h(x + y) = h(x) + h(y),$$

for all $x, y \in X$ and all $a \neq 0$. We say that $h, k : Y \rightarrow Y$ *commute* if

$$hk \subset kh.$$

For a linear relation $h : Y \to Y$, a subspace M of Y is called *h-weakly-invariant* if for all $x \in M$

$$h(x) \cap M \neq \emptyset.$$

The idea of the proof of our main theorem below (Corollary 6.5) can be traced back to the original Lomonosov's proof.

THEOREM 1.2. *Suppose Y is an infinite dimensional normed topological vector space, h and k are continuous linear relations with nonempty finite dimensional values, k is compact and commutes with h, $k^{-1}(0) \neq Y$. Then there is a nontrivial closed h-weakly-invariant subspace.*

Invariant subspace theorems for linear relations provide tools for studying the Invariant Subspace Problem for single valued linear operators. We consider those in the last section.

All topological spaces are assumed to be Hausdorff, all maps are multivalued with *nonempty* values unless indicated otherwise, by normed (locally convex) spaces we understand infinite dimensional normed (locally convex) topological vector spaces over $\mathbf{C}$ or $\mathbf{R}$.

2. Preliminaries.

Let X be a topological space. The a *partition of unity* is a collection of continuous functions $\gamma = \{d_\alpha : \alpha \in A\}$ satisfying

$$\sum_{\alpha \in A} d_\alpha(x) = 1, x \in X.$$

The partition γ is called *locally finite* if the cover $\gamma' = \{d_\alpha^{-1}((0,1]) : \alpha \in A\}$ of X is locally finite, and γ is called *subordinate to an open cover* ω of X if γ' refines ω.

Let $F : X \to Y$ be a multifunction (a set-valued map $F : X \to 2^Y$), where X, Y are topological spaces. We call F *lower-semicontinuous* (l.s.c.) if $F^{-1}(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$ is open for any open B. We call F *upper-semicontinuous* (u.s.c.) if $F^{-1}(B)$ is closed for any closed B. Equivalently, for any $x \in X$ and an neighborhood V of $F(x)$, there is a neighborhood U of x such that

$$y \in U \implies F(y) \subset V.$$

If F is both u.s.c. and l.s.c. we call it *continuous* (in [**2**] "continuous" means l.s.c.). When X and Y are uniform spaces [**3**, Chapter 8] we say that F is *uniformly upper-semicontinuous* (u.u.s.c.) if for any entourage V in Y there is an entourage U in X, such that for all $x, y \in X$,

$$x \in y + U \implies F(x) \subset F(y) + V.$$

We say that $x \in X$ is a *fixed point* of $F : X \to X$ if $x \in F(x)$. We say that a single-valued function $g : X \to Y$ is a *selection* of $F : X \to Y$ if $g(x) \in F(x)$ for all $x \in X$.

A topological space is called *acyclic* if its reduced Čech homology groups over the rationals are trivial. The set of all acyclic spaces includes convex subsets of locally convex spaces. A multivalued map $F : X \to Y$ is called *admissible* (in the sense of Gorniewicz [**4**]) if it is closed valued u.s.c. and there exist a topological space Z and two single-valued continuous maps $p : Z \to X$, $q : Z \to Y$ such that for any $x \in X$, (i) $p^{-1}(x)$ is acyclic, and (ii) $q(p^{-1}(x)) \subset F(x)$.

PROPOSITION 2.1. [**4**, Theorem IV.40.5, p. 200] *The composition of two admissible maps is admissible.*

In the proof of his theorem Lomonosov used the Schauder fixed point theorem. Here we need its analogue for multivalued maps. We use a result that follows from the Lefschetz fixed point theory for admissible maps given in [**4**] (for further developments see [**7**]), although an appropriate version of the Kakutani fixed point theorem for compositions of convex valued maps would suffice.

THEOREM 2.2. [**4**, Theorem IV.41.12, p. 207] *If X is an acyclic ANR, then X has the fixed point property within the class of compact (i.e., with $\overline{f(X)}$ compact) admissible maps.*

3. Approximation by Convex Combinations.

In this section we consider the approximations of multivalued maps by convex combinations in the spirit of Simonič [**8, 9**].

Let X be a topological space, Y be a vector space and H a collection of multivalued maps $h : X \to Y$. Given a partition of unity $\gamma = \{d_\alpha : \alpha \in A\}$ in X, for any collection $\{h_\alpha : \alpha \in A\} \subset H$, we can define a new map $h : X \to Y$ by

$$(3.1) \qquad h(x) = \sum_{\alpha \in A} d_\alpha(x) h_\alpha(x).$$

The set of all such h we denote by $Conv_\gamma(H)$.

Let Δ_n denote the standard n-simplex. For any collection $h_0, ..., h_n$ and any $(t_0, ..., t_n) \in \Delta_n$, we define a new map $h : X \to Y$ by

$$h(x) = \sum_{i=0}^{n} t_i h_i(x).$$

The set of all such h we denote by $Conv(H)$.

REMARK 3.1. $H \subset Conv(H) \subset Conv_\gamma(H)$.

The following theorem generalizes Lemma 3.1 of Simonič [**8**].

THEOREM 3.2 (Approximation). *Let X be a paracompact uniform space and Y be a locally convex space. Suppose $s : X \to Y$ is u.u.s.c. map with convex values and H is a collection of u.s.c. maps $h : X \to Y$. Suppose V is a convex neighborhood of 0 in Y and for any $x \in X$ there is $h_x \in H$ such that*

$$h_x(x) \subset s(x) + V.$$

Then there exist a locally finite partition of unity γ on X and a map $f \in Conv_\gamma(H)$ with values such that for all $x \in X$,

$$f(x) \subset s(x) + 3V.$$

PROOF. From the upper semicontinuity it follows that for each $x \in X$ there is a W_x such that for any $y \in W_x$,

$$h_x(y) \subset h_x(x) + V, \text{ and } s(x) \subset s(y) + V.$$

Now applying these inclusions and the assumption of the theorem, we obtain the following: for any $y \in W_x$,

$$h_x(y) \subset h_x(x) + V \subset s(x) + 2V \subset s(y) + 3V.$$

Consider the open cover of X given by $\omega = \{W_x : x \in X\}$. From Michael's Lemma [**3**, Theorem 5.1.9, p. 301] it follows that there exists a locally finite partition of unity $\gamma = \{d_\alpha : \alpha \in A\}$ subordinate to ω. Then we have a locally finite open cover of X

$$\omega' = \{U_\alpha = d_\alpha^{-1}((0,1]) : \alpha \in A\}$$

that refines ω, i.e., for each $\alpha \in A$ there is $x(\alpha) \in X$ such that $U_\alpha \subset W_{x(\alpha)}$.

Fix $y \in X$. Suppose $\alpha \in A$ and $d_\alpha(y) > 0$. Then $y \in U_\alpha \subset W_{x(\alpha)}$. Hence

$$h_{x(\alpha)}(y) \subset s(y) + 3V.$$

As $s(y)$ and V are convex, so is the set $s(y) + 3V$. Therefore a convex combination of $h_{x(\alpha)}(y), \alpha \in A$ is a well defined subset of $s(y) + 3V$. Then the map $f : X \to Y$ given by

$$f(y) = \sum_{\alpha \in A} d_\alpha(y) h_\alpha(y)$$

is well defined and belongs to $Conv_\gamma(H)$.   □

Simonič calls the functions d_α *Lomonosov functions* as the idea of this construction goes back to Lomonosov's proof in [**5**]. Observe also that this theorem implies the following well-known fact: *any u.s.c. map with convex images from a compact metric space to a locally convex space can be approximated by continuous single-valued maps.*

4. Fixed Points of Convex Combinations.

In this section we obtain a preliminary fixed point result.

PROPOSITION 4.1. *Let X be a topological space, Y a vector space, H be a collection of maps $h : X \to Y$, $r : Y \to X$ a map, γ a locally finite partition of unity on X, $f \in Conv_\gamma(H)$, and suppose $fr : Y \to Y$ has a fixed point y_0. Then there exists $g \in Conv(H)$ such that y_0 is a fixed point of gr.*

PROOF. We know that $y_0 \in fr(y_0)$. Let $Z = Graph(r) \subset Y \times X$, $p : Z \to y$, $q : Z \to X$ be the projections. Then there is $z_0 \in Z$ such that $y_0 = p(z_0) \in fq(z_0)$. Assume that $\gamma = \{d_\alpha : \alpha \in A\}$ and suppose

$$f(x) = \sum_{\alpha \in A} d_\alpha(x) h_\alpha(x),$$

where $h_\alpha \in H$. Let $x_0 = q(z_0)$. Suppose $\{\alpha \in A : d_\alpha(x_0) > 0\} = \{\alpha_0, ..., \alpha_n\}, n \geq 0$. For $i = 0, ..., n$, let

$$t_i = d_{\alpha_i}(x_0), h_i = h_{\alpha_i}.$$

Then $y_0 \in fq(z_0) = f(x_0) = \sum_{i=0}^{n} t_i h_i(x_0)$. We have an element of $Conv(H)$:

$$g(x) = \sum_{i=0}^{n} t_i h_i(x), x \in X.$$

Consider

$$
\begin{aligned}
gq(z_0) \quad &= \textstyle\sum_{i=0}^{n} t_i h_i(q(z_0)) \\
&= \textstyle\sum_{i=0}^{n} d_{\alpha_i}(x_0) h_i(q(z_0)) \\
&= \textstyle\sum_{\alpha \in A} d_{\alpha_i}(q(z_0)) h_i(q(z_0)) \\
&= fq(z_0) \\
&\ni p(z_0) = y_0.
\end{aligned}
$$

Thus y_0 is a fixed point of gr. $\square$

If Y is a topological vector space, we denote by $F_c(X, Y)$ the set of all u.s.c. maps $F : X \to Y$ with compact convex values. Then all elements of $F_c(X, Y)$ are admissible provided Y is locally convex.

LEMMA 4.2. $Conv_\gamma(F_c(X, Y)) \subset F_c(X, Y)$.

THEOREM 4.3 (Fixed Points). *Let Y be a locally convex space, $A \subset Y$ closed convex, U a closed neighborhood of A, X a paracompact uniform space. Let $r \in F_c(U, X)$, $\overline{r(U)}$ compact, $H \subset F_c(X, Y)$, and*

$$\overline{H(x)} \cap A \neq \emptyset, \text{ for all } x \in X.$$

Then for any neighborhood W of A, $Conv(H)r$ has a fixed point in W.

PROOF. Assume that there is a convex neighborhood V of 0 such that $U = A + 3V \subset W$ and for each $x \in X$, there is $h_x \in H$ such that $h_x(x) \cap (A + 1/2V) \neq \emptyset$. Now, for each $h \in H$, let

$$h'(x) = h(x) \cap \overline{(A + 1/2V)}$$

and let $H' = \{h' : h \in H\}$. Then the set $h'(x)$ is nonempty by assumption, convex since Y is locally convex, compact as the intersection of a compact set and a closed set. Also h' is u.s.c. by [**3**, 1.7.17(c)]. Thus $h' \in F_c(X, Y)$. Now we apply Theorem 3.2 with $s(x) = A$ for all x (by definition $h'_x(x) \subset s(x) + V$). Therefore there exists $f \in Conv_\gamma(H')$ such that for all x, $f(x) \subset s(x) + 3V = U$. We know that r, f are admissible. Therefore by Proposition 2.1, so is $\varphi = fr : U \to U$. Now r is compact and U is an ANR as a closed neighborhood in a locally convex space. Hence by Theorem 2.2, φ has a fixed point $y_0 \in U \subset W$. Therefore by Proposition 4.1, there exists $g \in Conv(H)$ such that y_0 is a fixed point of gr. $\square$

5. Properties of Linear Relations.

Throughout the rest of the paper we assume that X and Y are normed spaces.

DEFINITION 5.1. [**2**, II.1.3, p. 25] A multivalued map $h : X \to Y$ is called a *linear relation* if it preserves nonzero linear combinations, i.e., for all $x, y \in X$, all $a, b \in \mathbf{R} \setminus \{0\}$, we have

$$h(ax + by) = ah(x) + bh(y).$$

Then $h(0)$ is a linear subspace. The set of all linear relations will be denoted by $LR(X, Y)$, and $LR(X, X) = LR(X)$.

LEMMA 5.2. [**2**, Proposition I.2.8, p. 7] *If $T \in LR(X, Y)$, $x \in X$, then*

$$T(x) = y + T(0), \text{ for any } y \in T(x).$$

We will concentrate on the following classes of linear relations:

$$LR_0(X, Y) = \{h \in LR(X, Y) : h \text{ is continuous, } \dim h(0) < \infty\},$$
$$LR_0(Y) = LR_0(Y, Y).$$

Of course, all bounded linear operators belong to $LR_0(X, Y)$.

For a linear relation $T \in LR_0(X, Y)$, let Q_T denote the natural quotient map with domain X and null space $T(0)$ [**2**, p. 25].

LEMMA 5.3. *If $S \in LR_0(X, Y)$, $T \in LR_0(Y, Z)$, then $TS \in LR_0(X, Z)$.*

PROOF. $Q_T T : S(0) \to Z/T(0)$ is a linear operator, so $\dim Q_T TS(0) < \infty$. Now $\dim T(0) < \infty$ implies $\dim TS(0) < \infty$.   □

DEFINITION 5.4. [**2**] A linear relation $T \in LR_0(X, Y)$ is called *bounded* (*compact*) if the single valued operator $Q_T T$ is bounded (compact), i.e., it maps a bounded set into a bounded (compact) set.

By Proposition II.3.2(a) in [**2**, p. 33], every element of $LR_0(X, Y)$ is bounded.

LEMMA 5.5. *Suppose Y is normed. Then $T \in LR_0(X, Y)$ is compact if and only if for any bounded $B \subset X$, there is a compact set $C \subset Y$ such that $T(B) \subset C + T(0)$. Moreover C can be chosen such that $T(x) \cap C \neq \emptyset$ for all $x \in B$.*

PROOF. The "if" part is obvious. Next, if $Q_T T : X \to Z = Y/T(0)$ is a compact linear operator, then for any bounded $B \subset X$, there is a compact $D \subset Z$ such that $Q_T T(B) \subset D$. Now since $T(0)$ is a finite dimensional subspace of a normed space, it is topologically complemented, i.e., Y is homeomorphic to $Z \oplus T(0)$. Then $C = D \oplus \{0\}$ is compact in Y and $T(B) \subset C + T(0)$.   □

THEOREM 5.6. *Suppose $h \in LR_0(X, Y)$ and $k \in LR_0(Y, Z)$ is compact. Then $hk \in LR_0(X, Z)$ is compact.*

PROOF. Let A be a bounded subset of X. Then by Lemma 5.5, $k(A) \subset C + k(0)$, where C is compact. Therefore $hk(A) \subset h(C) + hk(0)$. It is easy to show that there is a bounded set D in Z such that $h(C) = D + h(0)$. Let $h'(x) = h(x) \cap \overline{D}$. Then $h'(x)$ is compact as $h(x)$ is finite dimensional. In particular, $h'(x)$ is closed, so by [**3**, 1.7.17(c)], $h' = h \cap \overline{D}$ is u.s.c.. Therefore $h(C) \cap \overline{D} = h'(C)$ is compact in Z by [**4**, Proposition II.14.9, p. 69]. But $D \subset h'(C)$, hence D is precompact. Thus

$$hk(A) \subset h(C) + hk(0) = D + h(0) + hk(0) = D + hk(0),$$

where D is precompact, so hk is compact by Lemma 5.5.   □

Therefore compact relations constitute a left ideal in $LR_0(Y)$.

We call $G \subset LR(Y)$ a *semialgebra* if it is closed under nonzero linear combinations and compositions. The above lemma implies that $LR_0(Y)$ is a semialgebra. We define the *commutant* of h as

$$Comm(h) = \{f \in LR_0(Y) : f \text{ commutes with } h\}.$$

LEMMA 5.7. *If $h \in LR(Y)$ then $Comm(h)$ is a semialgebra.*

PROOF. First, $Comm(h)$ is closed under linear combinations. Indeed, for $f, f' \in Comm(h)$, $a, b \in \mathbf{R} \setminus \{\mathbf{0}\}$, we have

$$h(af + bf') = ahf + bhf' \subset afh + bf'h = (af + bf')h.$$

Second, if f, g both commute with h, then $u = gf$ commutes with h. Indeed

$$hu = hgf \subset ghf \subset gfh = uh.$$

□

6. Invariant Subspaces of Linear Relations.

LEMMA 6.1. *Suppose $G \subset LR(Y)$ is a semialgebra. Then for any $u \in Y$, $G(u) \cup \{0\}$ is a linear subspace of Y.*

PROOF. Let $x, y \in G(u)$, then $x \in f(u), y \in f'(u)$ for some $f, f' \in G$. Suppose $a, b \in \mathbf{R} \backslash \{\mathbf{0}\}$ and let $g = af + bf' \in G$. Then

$$ax + by \in af(u) + bf'(u) = g(u) \subset G(u).$$

$\square$

LEMMA 6.2. *Suppose $R \in LR(Y)$. Then*

$$Fix(R) = \{x \in Y : x \in R(x)\}$$

is a linear subspace of Y.

PROOF. Let $a, b \in \mathbf{R} \backslash \{0\}, x, y \in Fix(R)$. Then $ax + by \in aR(x) + bR(y) = R(ax + by)$, so $ax + by \in Fix(R)$. $\square$

LEMMA 6.3. *Suppose $h \in LR(Y), R \in Comm(h)$ and $h(0) \subset R(0)$. Then $M = Fix(R)$ is h-weakly-invariant.*

PROOF. Let $x \in M = Fix(R)$. Then $A = h(x) \in hR(x) \subset Rh(x) = R(A)$. In particular, there is some $z \in A$ such that $R(z) \cap A \neq \emptyset$. Suppose $y \in R(z) \cap A$. Now we use Lemma 5.2, as follows:

$$z \in A = h(x) = y + h(0) \subset y + R(0) = R(z).$$

Hence $z \in Fix(R) = M$, so $h(x) \cap M \neq \emptyset$. $\square$

The main results of this paper are given below.

THEOREM 6.4 (Weakly Invariant Subspaces). *Suppose Y is a normed space, $h \in LR(Y)$, $G \subset Comm(h)$ is a semialgebra, $k \in Comm(h)$ is compact and $k^{-1}(0) \neq Y$. Suppose that for all $g \in G$, $h(0) \subset gk(0)$. Then there exists a nontrivial closed linear subspace $M \subset Y$ such that*

> *either (1) M is G-invariant,*
> *or (2) M is finite-dimensional and h-weakly-invariant.*

PROOF. For each $u \in Y$, $G(u)$ is G-invariant. Indeed, suppose $x \in G(u)$ and $g \in G$. Then $x \in f(u)$ for some $f \in G$. Therefore $gf \in G$ and $g(x) \subset gf(u) \subset G(u)$. Suppose now that $Q = G(u_0)$ is not dense in Y for some $u_0 \in Y \backslash \{0\}$. Then we can assume that $Q \neq \{0\}$, because otherwise $span\{u_0\}$ is G-invariant. Then $M = \overline{Q} = \overline{Q \cup \{0\}}$ is the desired subspace. First, $L = Q \cup \{0\}$ is a linear subspace of Y by Lemma 6.1. Second, since every $f \in G$ is u.s.c., $f(Q) \subset Q$ implies $f(\overline{Q}) \subset \overline{Q}$. Hence M is G-invariant.

Assume now that $G(y)$ is dense in Y for each $y \in Y \backslash \{0\}$. Since k is u.s.c., $k^{-1}(0)$ is closed. Therefore we can choose a closed convex neighborhood $U \subset Y$ of some $b \in Y \backslash \{0\}$ such that $0 \notin U$ and $U \subset Y \backslash k^{-1}(0)$. In addition we have

(*) $$b \in \overline{G(y)}, \text{ for all } y \in Y \backslash \{0\}.$$

Now we apply Theorem 4.3 with $X = Y \backslash \{0\}$, $A = \{b\}$ as follows. We let $H = \{g \cap U : g \in G\}$. Then $H \subset F_c(Y)$. We can also rewrite (*) as

$$\overline{H(x)} \cap A \neq \emptyset, \text{ for all } x \in Y \backslash \{0\}.$$

By Lemma 5.5, there is a compact $C \subset Y$ such that $k(x) \cap C \neq \emptyset$. Let $r = k \cap C \in F_c(Y, Y)$. Since k is compact, $r(U)$ is precompact. Therefore by Theorem 4.3, for any neighborhood W of b, $Conv(H)r$ has a fixed point in W. Therefore there is $g \in Conv(G) \subset G$ such that $R = gk$ has a fixed point $y_0 \neq 0$. Thus

$$M = Fix(R) \neq \{0\}.$$

Now $R \in Comm(h)$ (Lemma 5.7), M is a linear subspace of Y (Lemma 6.2), and M is h-invariant (Lemma 6.3).

Suppose now that $B \subset M$ is a bounded neighborhood of 0. Since R is compact by Theorem 5.6, we have $B \subset R(B) \subset C + R(0)$, where C is compact and $R(0)$ is finite dimensional. Therefore M is finite dimensional. $\qquad\square$

COROLLARY 6.5. *Suppose Y is a normed space, $h \in LR_0(Y)$, $k \in Comm(h)$ is compact and $k^{-1}(0) \neq Y$. Then there exists a nontrivial closed h-weakly-invariant subspace $M \subset Y$.*

PROOF. Let

$$G = \{P(h) : P \text{ is a polynomial without constant term}\}.$$

Then $G \subset Comm(h)$ and G is a semialgebra. To check the rest of the conditions of the theorem, observe that since $0 \in k(0)$, we have $h^n(0) \subset h^n k(0)$ for all $n \geq 0$. Therefore $h(0) \subset h^n(0) \subset h^n k(0)$ for all $n \geq 1$, so that $h(0) \subset P(h)(0)$ for any polynomial P without constant term. Thus for all $g \in G$, $h(0) \subset gk(0)$. Therefore by the theorem there is a nontrivial closed linear subspace $M \subset Y$ which is either G-invariant or h-weakly-invariant. Since $h \in G$, M is h-weakly-invariant. $\qquad\square$

REMARK 6.6. When both h and k are single valued, the corollary reduces to Lomonosov's Theorem 1.1. The corollary is vacuous when h is single-valued while k is not, because $hk \subset kh$ implies that $k(0)$ is a finite-dimensional h-invariant subspace. Yet in the next section we will obtain some applications of Theorem 6.4 to linear operators.

7. Applications to Invariant Subspaces of Linear Operators.

In this section we generalize a well known corollary to Lomonosov's Theorem 1.1.

LEMMA 7.1. *Suppose $h, k \in LR(Y)$. Then*

$$G = \{g \in Comm(h) : h(0) \subset gk(0)\}$$

is a semialgebra.

PROOF. By Lemma 5.7 $Comm(h)$ is a semialgebra, so we need only to consider the following. (1) Suppose $h(0) \subset fk(0)$ and $h(0) \subset gk(0)$, $a, b \in \mathbf{C} \backslash \{\mathbf{0}\}$. Then $h(0) \subset afk(0) + bgk(0) = (af + bg)k(0)$ because $h(0)$ is a linear subspace. Thus G is closed under nonzero linear combinations. (2) Suppose $h(0) \subset fk(0)$, g commutes with h. Now since $0 \in g(0)$, we have

$$h(0) \subset hg(0) \subset gh(0) \subset gfk(0).$$

Hence G is closed under compositions. $\qquad\square$

Suppose Y is a space over $\mathbf{C}$. Given $h \in LR(Y)$, its *eigenvalue* $\lambda \in \mathbf{C}$ and *eigenvector* $u \neq 0$ satisfy

$$\lambda u \in h(u)$$

(or $\ker(\lambda Id - h) \neq \{0\}$). The *eigenspace* of h corresponding to λ is given by

$$E_{h\lambda} = \{u \in Y : \lambda u \in h(u)\}.$$

LEMMA 7.2. *If $h \in LR(Y)$ is u.s.c. then $E_{h\lambda}$ is a closed linear subspace.*

PROOF. The set $E_{h\lambda} = (\lambda Id - h)^{-1}(0)$ is closed because h is u.s.c.. $\qquad\square$

The next lemma is obvious, see [**2**, Chapter 6].

LEMMA 7.3. *Let E be a finite dimensional space over $\mathbf{C}$. Then any $h \in LR_0(E)$ has an eigenvector.*

We say that the maps $h, k : Y \to Y$ *strictly commute* if

$$kh = hk.$$

THEOREM 7.4 (Invariant Subspaces). *Let Y be a normed space over $\mathbf{C}$, $h \in LR(Y)$ is u.s.c., for any $\lambda \in \mathbf{C}$, λId is not a selection of h, $k \in Comm(h)$ compact, $k^{-1}(0) \neq Y$. If a linear operator $f : Y \to Y$ strictly commutes with h and $h(0) \subset fk(0)$, then there is a nontrivial closed f-invariant subspace.*

PROOF. By Lemma 7.1

$$G = \{g \in Comm(h) : h(0) \subset gk(0)\}$$

is a semialgebra. Therefore by Theorem 6.4 there is a nontrivial closed subspace M that is either (1) G-invariant or (2) finite-dimensional and h-weakly-invariant. In case of (1) M is f-invariant because $f \in G$. Consider (2). If $h' = h \cap M : M \to M$ then h' is a linear relation with nonempty values. Then by Lemma 7.3, h' has an eigenvector corresponding to some $\lambda \in \mathbf{C}$. Therefore the eigenspace $N = E_{h\lambda}$ is nonzero, closed and not equal to the whole Y. Then for each $x \in N$, we have

$$\lambda f(x) = f(\lambda x) \in fh(x) = hf(x).$$

Hence $f(x) \in N$. $\qquad\square$

The following result involves only single valued operators.

COROLLARY 7.5. *Let Y be a normed space over $\mathbf{C}$, N be a finite dimensional subspace of Y, $f, h, k : Y \to Y$ bounded linear operators. Suppose for any $\lambda \in \mathbf{C}$, $h \neq \lambda Id$, k is nonzero compact. Suppose also that*

(1) $fh = hf$,
(2) $hk - kh \in N$,
(3) $h(N) \subset N$.

Then there is a nontrivial closed f-invariant subspace.

PROOF. Apply the above theorem to the linear relation $k + N$. $\qquad\square$

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